

Wheelock College Technology Course
Euclid's Algorithm: Debbie Borkovitz
May 2001

Explorations

For the explorations portion of the workshop, you can choose your own starting point. If you are new to Excel and/or you are nervous about the mathematics, I suggest you start with the tutorial. Otherwise, feel free to jump right to the suggested activities (or to suggest your own and do it...).

Tutorial:

1) First, do Euclid's Algorithm by hand (or using a calculator) to find the greatest common factor of 219 and 123. Write out every step. Your answer should be 3. Be sure you're clear on this step before you start using the computer.

2) Start up Excel. You will see a screen with lots of rectangular boxes. This is a worksheet. The columns are labeled with letters and the rows with numbers. Each rectangular box is called a *cell*. The cells are denoted A1, C17, etc. (C17 is a cell in column C, row 17 etc.). You can enter numbers, text, formulas, etc. into each cell.

3) In each row, we will represent one subtraction step in the algorithm. We don't actually have to type very much to make the table; most of it is automatic. We will place the larger number in column A, the smaller number in column B, and the difference in column C. Start labeling each column by typing something like "larger number" in cell A1. Don't type the quotes, and hit return when you are done. You probably will not have enough space, so you need to make column A wider. To do this, move the mouse to the gray area at the top of the worksheet, on the line between letters A and B. When you are in the right place, the cursor will change from a white cross to a black vertical line with horizontal arrows through it. Now you can drag to change the widths manually or double click to change them automatically. Label columns B and C by typing in cells B1 and C1.

4) We will start by having Excel do Euclid's algorithm using 219 and 123. Later, you can change a few values and the whole table will update automatically. Type 219 in cell A2 and 123 in cell B2.

5) Now, we want cell C2 to be 219-123. However, instead of typing the actual number, we will use a formula that references individual cells. In a short while, you'll see the power of using such a formula. In cell C2, start by typing an equal sign (=). Formulas always start with equal signs, and forgetting the equal sign is a common (easily corrected) mistake. Now either type A2-B2 or else click in A2 then type a minus sign and click in B2 (i.e. instead of typing cell references, you can just click in the cell – if you make a mistake doing this, type Esc and start over). Type return. You should see 96 in cell C2. Excel does the subtraction automatically.

6) Now we'll start on the second row of our table. This is the last row in which we'll have to type anything! Now, we need new numbers to go into cells A3 and B3 so we can compute the next row. We know that 96 is smaller than 123, however, it is possible for the number in C2 to be larger than the number in B2 (you might wish to

check your handwritten example for rows where the difference is larger than the smaller of the initial numbers). In cell A3, we wish to put the larger of B2 and C2. We don't know which one it will be, but we don't need to: Excel has a command called max that finds the maximum value of two numbers. In cell A3, type "=max(B2,C2)." Don't type the quotes or the period, and you can click in the cells instead of typing them. Hit return. Now use the analogous "min" command in cell B3. You should see the appropriate numbers in each cell.

7) Now click on cell C2. Notice that in the bottom right corner of the cell there's a little black box. This is called a *fill handle*. Move the cursor to this little black box. The cursor will change to a black cross. Hold down the left mouse button and drag the cursor down one cell. Let go. Cell C3 should show 27, which is A3-B3. Click on cell C3. If you look up above the worksheet, you will see the *formula bar*, which displays the actual formula used to calculate each cell. Note that the formula has changed automatically from A2-B2 to A3-B3 (also click in C2 to see the difference).

8) Now comes the fun part! Click in cell A3 hold down the mouse button and drag to cell C3. You should see an outline around all three cells, with the right two shaded. Note the fill handle at the bottom right corner. Drag this fill handle down as many rows as you wish (at least 10). You should see an entire table, pretty much like the one you did by hand. Click in various cells, look at the formula bar, and see how Excel has changed the formulas automatically.

9) Now comes the other fun part. Change the numbers in A2 and B2. The whole table should change automatically. You might have to drag the formulas down more cells, but otherwise, you now have a program for executing Euclid's algorithm for any two natural numbers.

10) If there's time, go on to one of the suggested activities below.

Suggested Activities (Start wherever you wish):

1. Use multiplication and division to reformulate Euclid's Algorithm and make it less tedious.
2. Design a spreadsheet that implements Euclid's Algorithm (either version). If you already did the tutorial, find a different way to implement the spreadsheet. Some commands you might find useful include: int, mod, min, max.
3. Explore other things to do with Excel.
4. Bezout's theorem is important in Number Theory. It states that if d is the greatest common divisor of a and b , then there are integers x and y such that $ax+by=d$. For example, the greatest common divisor of 219 and 123 is 3 and $3 = 219 \cdot 9 - 123 \cdot 16$. It is not obvious how to find x and y , but (as you might guess), Euclid's Algorithm (done kind of backwards) makes it much easier. Explore Bezout's theorem. Can you find integers x and y so that $75x + 392y = 1$? Can you find a way to use a spreadsheet as a tool in solving equations such as this one? (By the way, equations whose solutions must be integers are called *Diophantine Equations*).

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Background

The Distributive Property:

The distributive property is key to why Euclid's Algorithm works. Below it is illustrated with a specific example. Think about *how* to generalize this example.

3 is a *factor* of 18 (or we say 3 *divides* 18) because $18 \div 3$ has no remainder.

Visually, we can represent 18 as groups of 3:



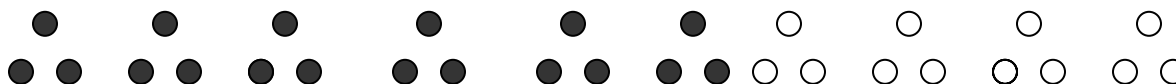
As an equation, we have $18 = 6 \cdot 3$.

Now, 3 is also a factor of 12, because $12 \div 3$ has no remainder. Visually we have,



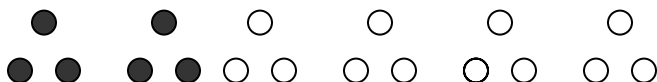
and as an equation we have, $12 = 4 \cdot 3$.

Now, the distributive property tells us that 3 is a factor of both $18+12$ and of $18-12$. Visually, we see that $18+12$ can be written as $6+4=10$ groups of 3:



As an equation, we have $18+12 = 6 \cdot 3 + 4 \cdot 3 = (6+4) \cdot 3 = 10 \cdot 3$.

Similarly, $18-12$ can be written as $6-4=2$ groups of 3:



As an equation, we have $18-12 = 6 \cdot 3 - 4 \cdot 3 = (6-4) \cdot 3 = 2 \cdot 3$.

For understanding Euclid's Algorithm, the important consequence of the distributive property is that if f is a factor of both a and b , then f is a factor of both $a+b$ and of $a-b$.

An Illustration of Euclid's Algorithm:

We will find the greatest common factor of 72 and 30. The *greatest common factor* is exactly what it sounds like: it is the largest natural number (i.e. whole number) that is a factor of both 30 and 72.

By the distributive property, we know that any factor of both 72 and 30 is also a factor of $72-30=42$.

Similarly, any factor of both 30 and 42 is also a factor of $42-30=12$ (and thus, any factor of both 72 and 30 is also a factor of 12).

Continuing in this manner, any factor of both 30 and 12 is also a factor of $30-12=18$.

Any factor of both 18 and 12 is also a factor of $18-12=6$.

Any factor of both 12 and 6 is also a factor of $12-6=6$.

Now we are left with 6 and 6. Following the entire chain of reasoning above, any factor of both 72 and 30 must also be a factor of 6.

Now, the greatest factor of 6 is 6 itself. So, if we can show that 6 must be a factor of 30 and 72, then 6 must be the greatest common factor of 30 and 72. Of course, 6 is a factor of both 30 and 72, but to use this example to illustrate how to prove that Euclid's Algorithm works in general, we note that we can use the distributive property looking at the algorithm *backwards*. Since 6 is a factor of both 6 and 6, it is a factor of $6+6=12$. Since 6 is a factor of 6 and of 12, it is a factor of $6+12=18$, etc. Continuing this way, we see that 6 is a factor of both 30 and 72, hence it is the greatest common factor.

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