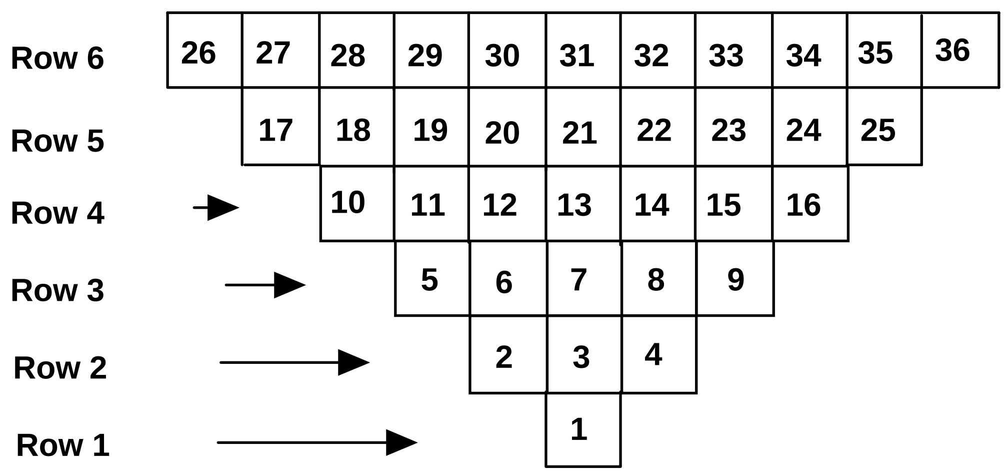
Characteristics of Linear, Quadratic, and Exponential Functions:
Some Examples

This handout explores linear, quadratic, and exponential functions through a few examples. The treatment is not designed to be complete, but rather to give an accessible introduction to some functions that commonly arise in problem-solving activities. Such functions can be represented by tables, graphs, recursive and explicit equations, as well as by verbal and pictorial descriptions -- flexibility in moving between such representations is a key algebraic thinking skill.

Linear Functions:

Example: A Stadium Problem

The first six rows of a stadium section are shown below. The rows continue the same pattern, and there are 50 rows total.



How many seats are there in Row R?

We shall let S represent the number of seats in row R. We can make the following table of R and S:

R = row number	S = Number of Seats in Row R
1	1
2	3
3	5
4	7
5	9
6	11
7	13

As the row number increases by 1, the number of seats increases by 2. This pattern is implicit in the way the problem was designed – each row of seats contains two more than the one before it.

One way to characterize linear functions is that they can be written in table form so that the differences in both the input and output columns are constant. Sometimes in problem solving, it's helpful to mark the differences explicitly:

	R = row number	S = Number of Seats in Row R	
	1	1	
2-1=1			3-1=2
	2	3	
3-2=1			5-3=2
	3	5	
1			2
	4	7	
1			2
	5	9	
1			2
	6	11	
1			2
	7	13	

Note that in most situations, a table only includes some of the values of a function. The characterization says that a linear function <u>can</u> be written as a table in this form, but if we include different values of the function, the table might seem to have different properties. Here are some other examples of tables of values taken from this same function:

	R	S	
	1	1	
2			4
	3	5	
2			4
	5	9	
2			4
	7	13	
2			4
	9	17	

	R	S	
	1	1	
5			10
	6	11	
5			10
	11	21	
5			10
	16	31	
5			10
	21	41	

	R	S	
	5	9	
-2			-4
	3	5	
17			34
	20	39	
10			20
	30	59	
-19			-38
	11	21	

Notice that in the first two tables, the input and output differences clearly remain constant, but they are different constants than in the original example. The last table is unorganized, and the pattern is not clearly visible, although the values are all accurate for rows of the stadium problem.

You might have noticed another pattern that holds for all three tables: the increase (or decrease) on the right side of the table is always double the increase (or decrease) on the left side of the table. In the rightmost table, for example, the input (R.) column first decreases by 2, and the output (S) column decreases by 4, and 4 is twice 2. Then the R column increases by 17, and the S column increases by 34, or twice 17.

The ratio of the increase in the output column to the corresponding increase in the input column is called the *slope* of the function. Traditionally, the slope concept was introduced in graphical and explicit equation representations of linear functions, but for many students, this concept is most intuitive when presented in the table representation of a function.

Exercises (answers are in the back):

1. Determine whether each of the following tables represents a linear function (if the values in the table are consistent with a linear function, you may assume that the whole function is linear – i.e. that the function doesn't behave differently on values not represented in the table).

a.

x	y
0	4
1	9
2	14
3	19
4	24
5	29

b.

a	b
0	17
3	11
6	5
9	-1
12	-7
15	-13

c.

in	out
0	2
1	4
2	8
3	16
4	32
5	64

d.

p	q
7	22
2	7
3	10
0	1
5	16
6	19

2. Find the slope of each of the linear functions in part 1.

Recursive Equation Representation:

A *recursive equation* (also called a *recurrence relation*) is an equation whose next value depends on previous values. A recursive equation is a natural representation of the stadium problem. We can describe the number of seats in row R as follows:

The number of seats in Row 1 = 1
The number of seats in Row R = The number of seats in Row (R-1) + 2

Giving the number of seats in Row 1 is an example of an *initial condition*. We need to know how to start our table. Once we have an initial condition, we can create the next row from the previous row.

Recursive equations and tables can be highly inefficient methods for solving problems – in order to find the number of seats in row 50, using either representation, we must make a table or compute values from rows 1 to 49 first. With technology, however, recursive equations can become efficient calculating tools; they are easily entered in both spreadsheets and most graphing calculators, which can then be used to quickly compute large values. Thus, many topics can become accessible to students who have not yet mastered algebraic notation or graph interpretation (and recursive equations and tables can be used to make connections to help students understand other representations).

In Excel, we can enter the above recursive equation by first typing a 1 in cell A1, to represent the initial condition. Then, in cell A2, we can type "=A1+2," which will add 2 to cell A1 to compute the value in cell A2. Then, if we drag the formula down from cell A2, it will automatically update, and we can easily drag it down 50 cells to compute the number of seats in any row of the stadium.

More formal mathematical notation for the recursive equation looks like this:

$$S(1) = 1; \quad S(R+1) = S(R) + 2$$

or

$$S_1 = 1; \quad S_{R+1} = S_R + 2$$

This notation can seem intimidating, but it is almost identical to description with words given above.

When represented as an explicit equation, a linear function takes the following form:

Initial Row = ____ (fill in blank with a number)

Next Row = Previous Row + ___ (fill in blank with a number)

More formally,

$$S(0) = a; \quad S(R+1) = S(R) + b,$$

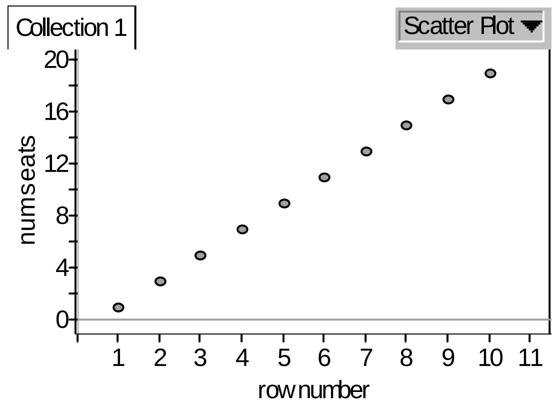
where a and b can represent any numbers (including negative numbers, so the second condition can look like subtraction instead of addition; also we can have $a = b$). Note that the above representation starts with $S(0)$, so that the initial condition above is given to start at input = 0, but we can actually start anywhere.

Exercise:

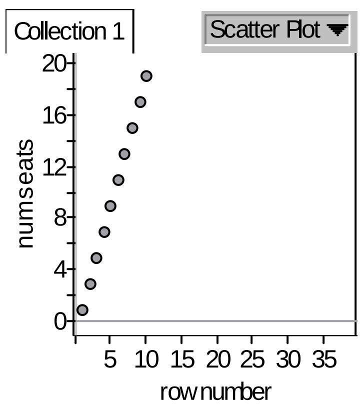
3. Represent each of the functions in problem 1 as a recursive equation (even if the function isn't linear). Explain how to tell from the recursive equations whether the function is linear.

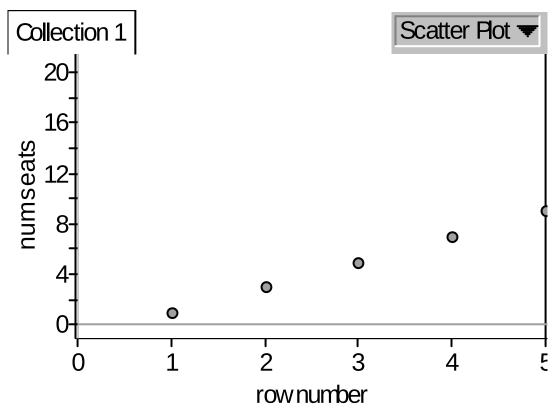
Graph Representation

A linear equation is called *linear* because its graph is a straight line. In the case of the stadium problem, the graph is actually a series of points that are on a straight line – the function that determines the number of seats in each row is not defined if the rows are not natural numbers (i.e. it's defined if the row number is 1, 2, 3, 4, etc. but not if the row number is 1/2 or –6).



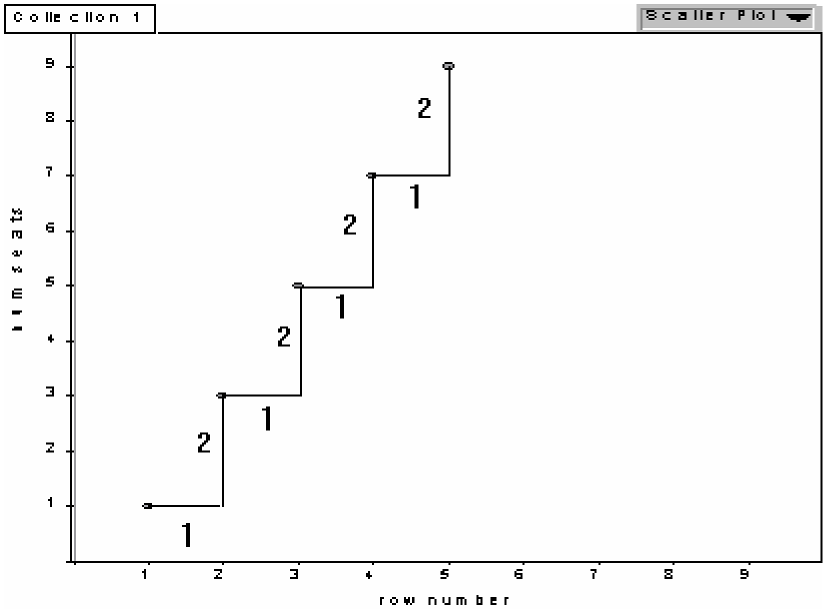
Note that according to the everyday definition of *slope* (i.e. the amount a line is slanted), this graph can appear to have different slopes, depending on how the axes are scaled:





Both of the above graphs represent the data in the staircase problem. There is no one "right" way to scale the axes; what's most important is to recognize how scaling can make graphs misleading.

As it is defined mathematically, the slope represents the vertical increase when the horizontal value increases by 1. We can see on the graph, as we did in the table, that when the row number (horizontal value) increases by 1, the number of seats (vertical value) increases by 2. This relationship remains true no matter how the graph is scaled (although it can be harder to detect accurately depending on the graph).



Explicit Equation Representation:

Most traditional high school algebra courses focus primarily on manipulating explicit equations (which they often simply refer to as "equations"). An explicit equation is an enormously efficient way to code a lot of information.

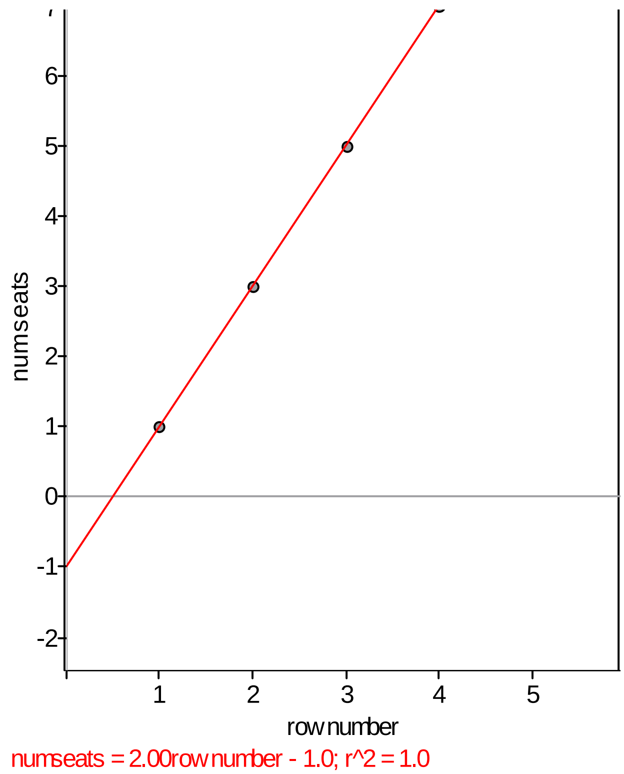
To find our explicit equation, we will start by extending the graph to a straight line, i.e. by extending the domain (set of possible input values) to include all real numbers (i.e. all numbers on the number line). Not all points on this line will represent rows of the stadium, but our equation will represent all points on the line (i.e. we will be able to, for example, substitute 1/2 for a row number, and although the answer won't tell us anything about the stadium problem, it will give us a point on the line).

Note that the graph (see next page) starts at rownum=0, and for rownum=0, we have numseats= -1. It doesn't make much sense to say that row 0 has –1 seat in it, but if we keep extending the table or graph according to the same pattern, it makes sense that –1 would correspond to 0 in this way. The value –1 is known as the *y-intercept*; it is the place where the graph crosses the vertical axis.

Here's how looking at the y-intercept can help us. Let's start with an example. Suppose we want to find the number of seats in row 15. We will start with the mythical row 0, where there is –1 seat. Now, to get to row 15 from row 0, we move 15 rows (in the stadium itself, we are moving back; on the graph, we are moving right), and we know that for every row we move, we will add two seats, so that we will add 30 seats to our original –1 seat, for a total of 29 seats in row 15. You can use any of the other representations to check that this answer is correct.

If the –1 makes you uncomfortable (and it very well might), we can instead start at row 1, with 1 seat, and move 14 rows. In moving 14 rows, we will add 28 seats, for a total of 29 seats.

Now, to find the number of seats in row r, we can start with row 0 and move r rows. Following the logic of the example, we see that number of seats in row r = -1 + 2 r (remember that 2 r means 2 times r)



Following the logic of the second example, we see that

number of seats in row r = 1 + 2 (r-1), and by the distributive property, we have
number of seats in row r = 1 + 2r -2 = -1 +2r (just as before)

In algebra, you might remember describing linear equations in the form $y = mx + b$. In this notation, b represents the y intercept (-1 in our case), m represents the slope (2 in our case), x represents the input or horizontal values (the row number in our case), and y represents the output or vertical value (the number of seats in row r).

Exercises:

4. For each of the following linear functions, provide the missing representations (table, graph, explicit, and or recursive equation).

a. $y = 5x - 3$

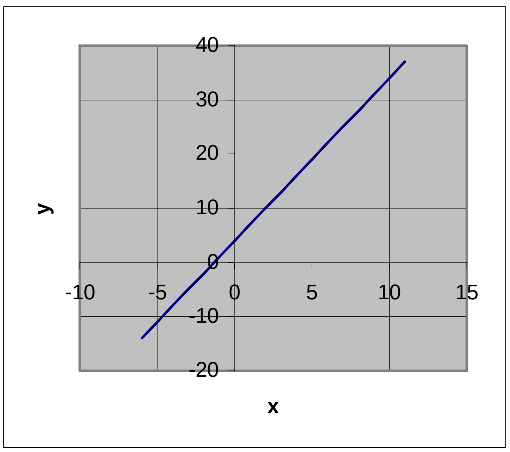
b. $S(n+1) = S(n) + 2; \quad S(0) = 11$

c. A taxi cab fare costs $1.50 when you step into the cab and $.25 for each 1/8 of a mile that you go.

d.

X	Y
-9	20
-6	17
-3	14
0	11
3	8
6	5
9	2

e.



Quadratic Functions:

Quadratic functions are more difficult than linear functions, and we will not describe them in as much detail in this handout. Even a little knowledge about quadratic functions can be helpful in problem solving, however. We will look at three more problems related to the stadium seating; these problems can be described by quadratic, rather than linear, functions.

- What is the seat number of the last seat in row r?
- What is the seat number of the middle seat in row r?
- What is the seat number of the first seat in row r?

We begin with the first question. A reasonable start is to make a table:

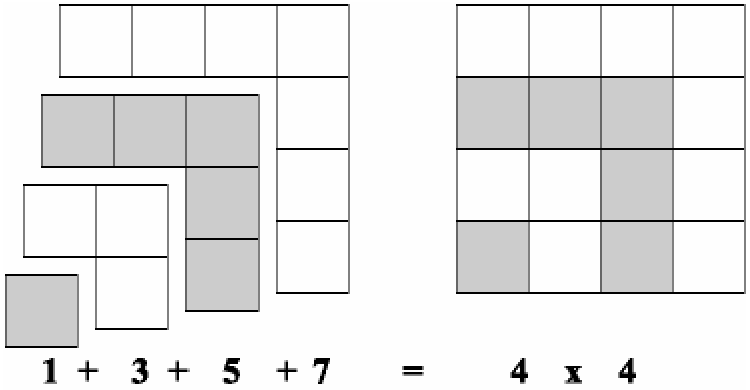
R = row number	L = Number of last seat
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64

As before, we can make a difference table:

R = row number		L = Number of last seat	
	1	1	
2-1=1			4-1=3
	2	4	
3-2=1			9-4=5
	3	9	
1			7
	4	16	
1			9
	5	25	
1			11
	6	36	
1			13
	7	49	

Note that in this case, the differences on the right form a linear function (as opposed to when we looked at linear functions, and the differences were all constant). When the difference table is equally spaced on the input side, with differences that form a linear function on the output side, the function is quadratic.

We won't go into quadratic functions in depth, but one important thing to know about them is that when represented as explicit equations, they include a term that is squared (i.e. multiplied by itself). In fact, looking more carefully, we can easily see that $L = R \times R = R^2$, e.g. the last seat in row 6 is 6 x 6 = 36. The picture below demonstrates why the differences in this table form a linear function:



If we compare the 3 x 3 square and the 4 x 4 square visually, we see that the difference forms the rotated L shape, and this L shape includes two more square tiles than the previous difference, which corresponds to the differences in the table increasing by 2 each time..

Notice that the differences form the same linear function that we defined in the first stadium problem, i.e. 2r-1. We can use this information to write L, the number of the last seat in row r, of the stadium problem as a recursive function:

$$L(1) = 1; \quad L(r+1) = L(r) + 2r - 1$$

When represented as an explicit equation, a quadratic function takes the following form:

Initial Row = ____ (fill in blank with a number)

Next Row = Previous Row + ___ (fill in blank with a linear function)

More formally,

$$S(0) = a; \quad S(R+1) = S(R) + bx + c\ .$$

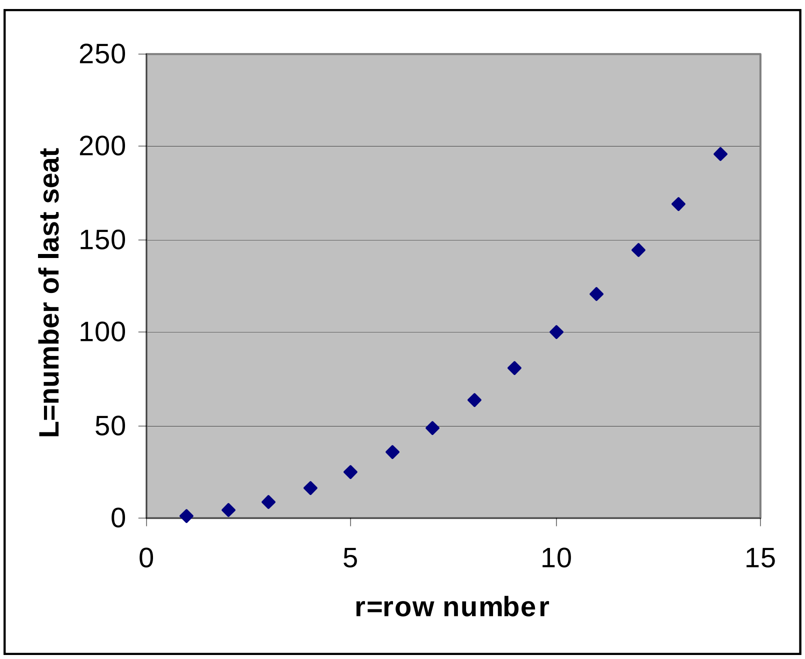
The a represents the initial condition, and $bx + c$ represents the general form of a linear function. The variables a and c can be any real numbers (including negative numbers and 0), but the variable b cannot equal 0, otherwise the recursive equation would represent a linear function, not a quadratic function.

We can use the recursive equation for the last seat in the row to set up a table in Excel as follows (formulas used in row 4 of the table are listed below the variables; the numbers in row 3 of the table are initial conditions and must be entered explicitly):

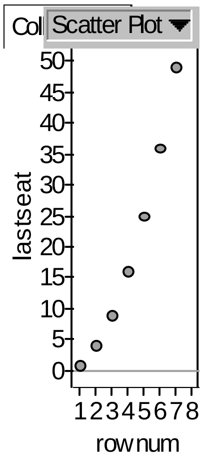
r=Row Number (A3+1)	S= Number of Seats in Row (B3+2)	L=Last Seat Number (B4+C3)
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64
9	17	81
10	19	100
11	21	121
12	23	144
13	25	169
14	27	196

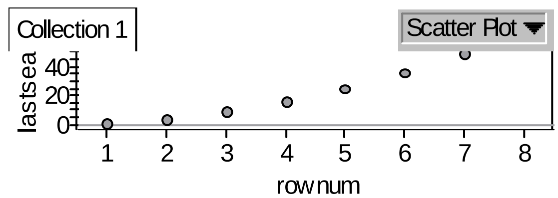
Although the explicit equation is very simple for this function (perfect squares), many students notice the recursive equation well before they notice the explicit equation.

Here is a graph of the last seat in each row as a function of the row number:



Once again, the scale of the function affects how its graph looks:





The last graph looks almost like a straight line – one of the key concepts in calculus is looking at curves at a scale where they resemble straight lines.

The graphs of quadratic functions are called *parabolas*. It is not a good idea to try to identify parabolas visually, as there are many other functions whose graphs look similar.

Now let's see how we might solve a more difficult quadratic problem, namely finding M, the seat number of the middle seat in row r. Once again, we can begin with a table, with differences included:

R = row number L = Number of last seat
R = row number M = Number of middle seat

Difference	R	M	Difference
	1	1	
2-1=1			3-1=2
	2	3	
3-2=1			7-3=4
	3	7	
1			6
	4	13	
1			8
	5	21	
1			10
	6	31	
1			12
	7	43	

We can write the following recursive function:

$$M(1) = 1; \quad M(r+1) = M(r) + 2r$$

and set up an Excel spreadsheet:

r=Row Number (A3+1)	Differences (B3+2)	M=Middle Seat Number (B4+C3)
1	0	1
2	2	3
3	4	7
4	6	13
5	8	21
6	10	31
7	12	43
8	14	57
9	16	73
10	18	91
11	20	111
12	22	133
13	24	157
14	26	183

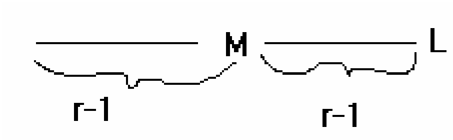
Note that the differences are shifted slightly from the last problem (why)?

To find an explicit equation, we can start by noticing from the difference table that the function is quadratic, which means that the explicit equation must involve squaring the row number. One method is to play around with multiplying r by something close to r. If we are very clever, we might notice that $r \cdot (r-1)$ is very close to M:

R = row number	R x (R-1)	M = Number of middle seat
1	1x0=0	1
2	2 x 1 =2	3
3	6	7
4	12	13
5	20	21
6	30	31
7	42	43

Thus, we can find a possible explicit equation as: $M = r(r-1)+1$. Note that we have done nothing to justify that this is the correct equation for this problem. Also note that being able to use the table to determine that the function must be quadratic makes it much easier to guess at an explicit function.

Another method to find an explicit equation is to start from L, the last number in each row, which is a perfect square, and use S, the number of seats in row r, to work backward to find M. We know that there are 2r-1 seats in row r. Without the last seat, there are (2r-1)-1 = 2r-2 =2(r-1) seats in row r, which are arranged as follows:



Thus, we can see that $L-(r-1)=M$ or $M=r^2-(r-1)$. If you are comfortable with algebraic manipulation, you can easily check that our two explicit equations, $M=r(r-1)+1$ and $M=r^2-(r-1)$, are equivalent. If you are not comfortable with manipulation, you can enter each equation in a spreadsheet and see that they agree for many values. We have used the problem situation to justify that the second equation is correct. Often in mathematics, we use intelligent guess-and-check to find a solution and then return to the problem to find another solution that we can prove.

Exercises:

5. Make a table, find a recursive and an explicit equation for F, the seat number of the first seat in row r.

6. Suppose we have a stadium that is similar to the one above, but with three seats in the first row, five in the second row, seven in the third row, etc. Find equations (or other representations) for the first, middle, and last seats in each row.

Exponential Functions:

We begin with the following sample problem:

- Suppose we start with a piece of paper, with area 64 sq. inches, and we begin folding it. Assume we are living in mathland, where we can keep folding forever without the folds actually taking up any space, and that the folded paper pieces line up perfectly. How many layers of paper are there after f folds? What is the area of the top layer?

Once again, we can start by making a table. We define N as the number of layers after f folds and A as the area of the top layer after f folds. The number of layers doubles with each fold, and the area of the top layer is halved with each fold.

f = number of folds	N = num. layers	A=Area (sq inches)
0	1	64
1	2	32
2	4	16
3	8	8
4	16	4
5	32	2
6	64	1
7	128	.5

For exponential functions, when the inputs increase by a constant amount, the outputs increase or decrease by the *same ratio*, as seen in the tables below:

	f = num folds	N = Number of layers	
	0	1	
1-0=1			2/1 = 2
	1	2	
2-1=1			4/2 = 1
	2	4	
1			8/4=2
	3	8	
1			16/8=2
	4	16	
1			32/16=2
	5	32	
1			64/32=2
	6	64	

	f = num folds	A = Area (sq inches)	
	0	64	
1-0=1			32/64 = .5
	1	32	
2-1=1			16/32=.5
	2	16	
1			8/16=.5
	3	8	
1			4/8=.5
	4	4	
1			.5
	5	2	
1			.5
	6	1	

Note that with an exponential function, if we change the spacing of the inputs, we will change the ratio, although the ratios will still be the same as each other, as long as inputs are equally spaced. For example:

	f = num folds	N = Number of layers	
	0	1	
3-0=3			8/1 = 8
	3	8	
6-3=3			64/8=8
	6	64	
3			512/64=8
	9	512	
3			4096/512=8
	12	4096	

The ratios do not have to be integers. For example, suppose we invest $1000 at 10% annual interest, compounded once a year. Here is a table of the amount of money that we will have at the end of each year:

	t = number of years	B = balance in account	
	0	$1000	
1-0=1			1100/1000 = 1.1
	1	$1100	
2-1=1			1210/1100 = 1.1
	2	$1210	
1			1331/1210=1.1
	3	$1331	

The ratio 1.1 is equivalent to 110% or 100% of the previous balance, with 10% of the previous balance added to it.

The pattern in the table of an exponential function resembles the pattern in a linear function, in that it can be checked easily as long as the input values are equally spaced. The constants in an exponential function come from division, as opposed to subtraction. Recall that when the input values have difference one, then the constant difference in a linear function is called the slope. Analogously, when the differences in the input values of an exponential function have difference one, then the constant ratio in an exponential function is called the *base*. Our function N, the number of layers, has base 2; the function A has base .5, and the function B has base 1.1. If the base is greater than one, then function is called *exponentially increasing*; if the base is less than one, the function is called *exponentially decreasing*. The base cannot be 0 or 1; if it were the function would be a constant, not an exponential function. Negative bases can get complicated.

7. Determine whether each of the following tables represents an exponential function (if the values in the table are consistent with an exponential function, you may assume that the whole function is exponential – i.e. that the function doesn't behave differently on values not represented in the table).

a.

x	y
0	12
1	24
2	48
3	96
4	192
5	384

b.

a	b
0	50
1	5
2	.5
3	.05
4	.005
5	.0005

c.

in	out
0	1
1	3
2	7
3	15
4	31
5	63

d.

p	q
0	.06
1	1.2
2	24
3	480
4	9600
5	192,000

8. For each of the above functions that is exponential, find the base.

Representation as a Recursive Equation

We can define the representation as a recursive equation similarly to that of linear functions. For N, the number of layers after f folds we have:

The number of layers after 0 folds = 1
The number of layers after f+1 folds = 2 x (the number of layers after f folds)

More formally, we have:

$$N(0) = 1; \quad N(f+1) = 2 \cdot N(f)$$

or

$$N_0 = 1; \quad N_{f+1} = 2 \cdot N_f$$

For A, the area in square inches after f folds we have:

The area after 0 folds = 1
The area after f+1 folds = .5 x (the number of layers after f folds)

More formally, we have:

$$A(0) = 1; \quad A(f+1) = .5 \cdot A(f)$$

or

$$A_0 = 1; \quad A_{f+1} = 2 \cdot A_f$$

For B, the bank balance in dollars after y years at 10% interest, we have:

The balance after 0 years = 1
The balance after y+1 folds = 1.1 x the balance after y years

More formally, we have:

$$B(0) = 1000; \quad B(y+1) = 1.1 \cdot B(f)$$

or

$$B_0 = 1000; \quad B_{y+1} = 1.1 \cdot B_y$$

When represented as an explicit equation, an exponential function takes the following form:

Initial Row = ____ (fill in blank with a number)

Next Row = Previous Row x ___ (fill in blank with a number)

More formally,

$$S(0) = a; \quad S(R+1) = b \cdot S(R),$$

where a can represent any number and (for now at least) b can represent any positive number besides 1.

Exercise:

9. Represent each of the functions (exponential or not) in problem 7 with a recursive equation.

Representation as an Explicit Equation:

The logic of representing exponential functions as explicit equations also resembles the logic of representing linear functions as explicit equations. Recall that with linear functions, we started with the value of the output when the input was 0, and then we successively added the slope. For exponential functions, we again start with the output value when the input is 0, but we *multiply* successively by the base.

For example, suppose we want to find the number of layers after 6 folds. We start with 1 fold, and then double it six times, to get:

$$1\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 = 1\cdot 2^6 = 2^6 \text{ layers.}$$

More generally, if we folded x times, we would have

$$N = 1\cdot 2^f = 2^f \text{ layers}$$

The "1" has been included in the above equation to make the format clearer. Contrast with the explicit equation for B, the balance after y years:

$$B = 1000\cdot(1.1)^y .$$

The explicit equation for A is

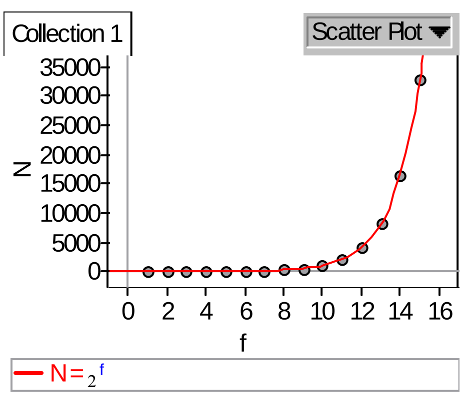
$$A = 1\cdot(.5)^f = (.5)^f .$$

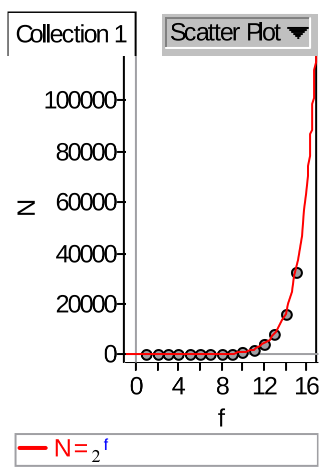
Exercise:

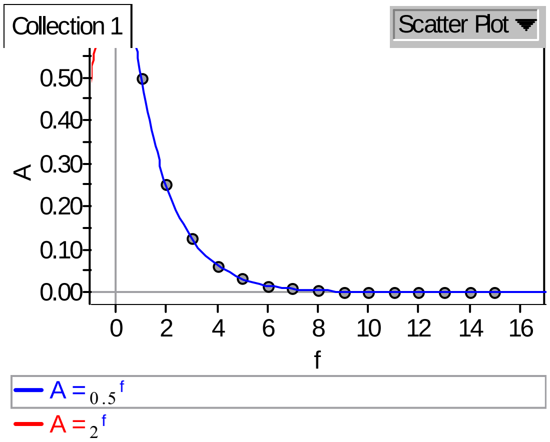
10. Represent each of the exponential functions in problem 7 with an explicit equation.

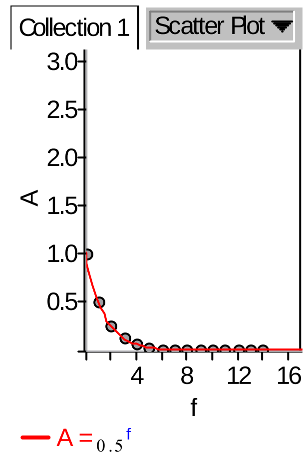
Graph Representation:

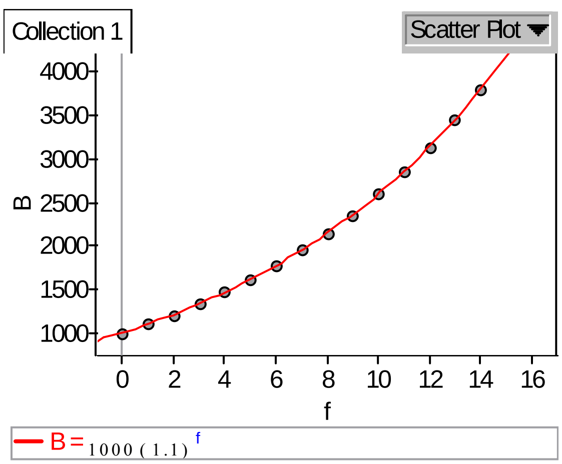
Graphs of exponential functions eventually grow or decrease really, really quickly, and many people do not have good intuition about them. As we discussed earlier, with only 15 folds, paper can reach the ceiling, and in 35 years, \$1 invested at 10% interest will grow (approximately) to \$32. Here are the graphs of the functions we've discussed, with each represented at two different scales:

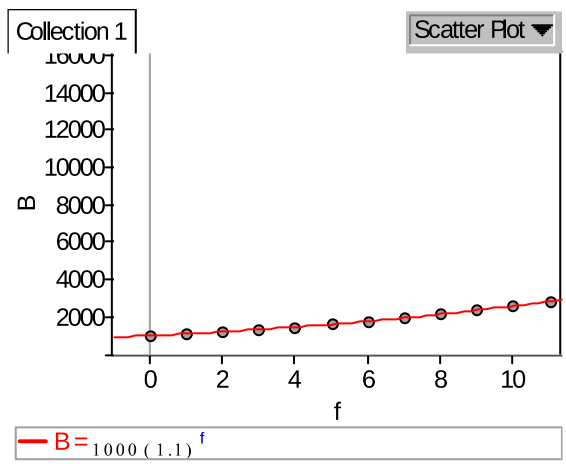


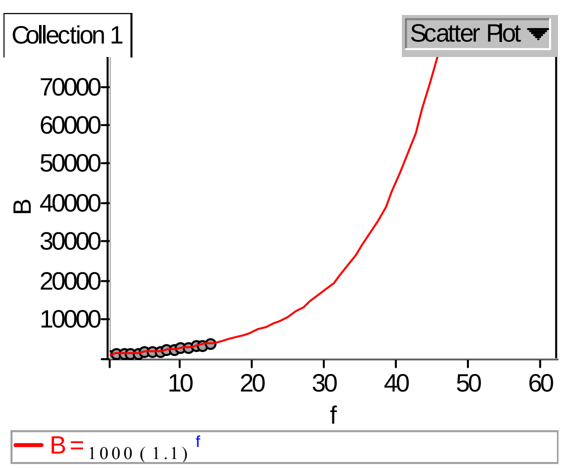












Although exponential functions can look very different at different scales, they also have the property that they can look very similar at very different scales. Note how similar are the shapes of the first and third graph for the balance, but how different are the scales.

Answers to Exercises

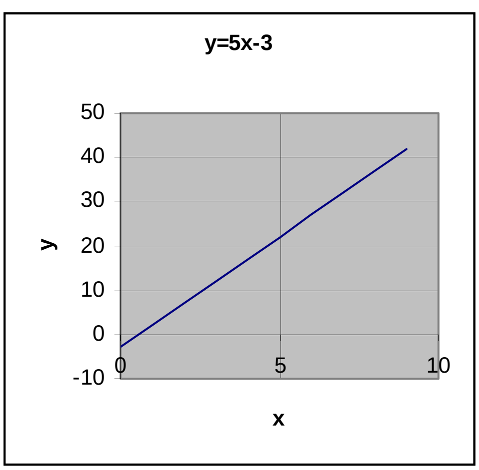
1. and 2. a. linear slope 5 (as x increases by 1, y increases by 5). b. Linear, slope –2 (as a increases by 3, b decreases by 6 and the slope is –6/3= -2; the negative sign indicates decrease, and as a increases by 1, b decreases by –2). c. Not linear. This is an example of an *exponential* function. As input increases by 1, output doubles, which is different from increasing by a constant amount. d. Linear. You can rearrange the tables and fill in missing values. The slope is 3 – as p increases by 1, q increases by 3.

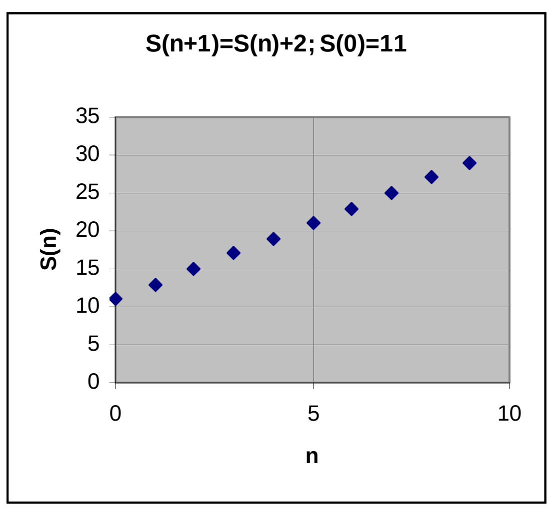
3. a) y(0)=4 and y(r+1)= y(r)+5 b) b(0)=17 and b(r+1)=b(r)-2 c. out(0)=2 and out(r+1) = 2 * out (r) (note that this isn't the form of the linear function, we are multiplying by 2, not adding the same thing every time). d. q(0)=1 and q(r+1)=q(r)+3.

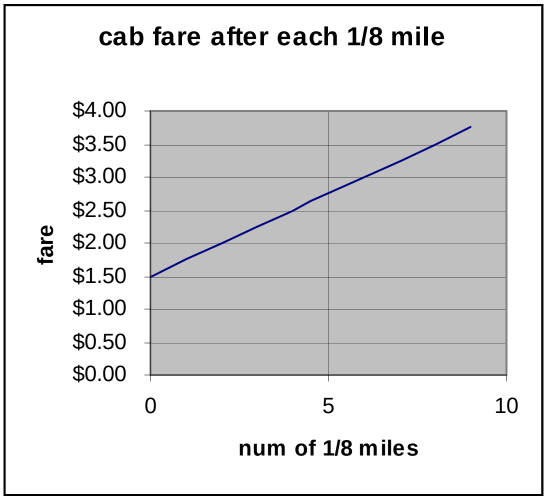
4.

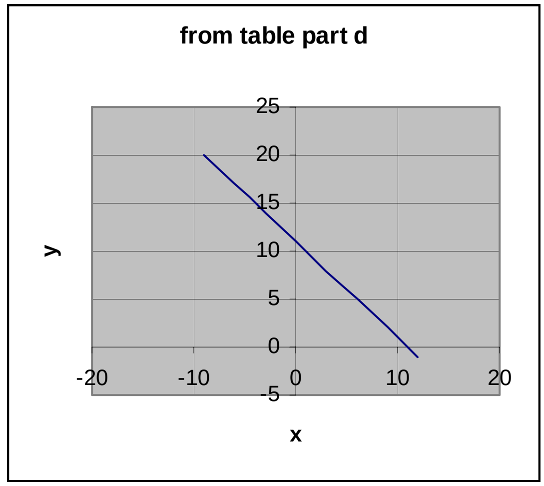
input	a. y=5x-3	b. S(n+1)=S(n)+2	c. cab fare after each 1/8 mile	e. from graph
0	-3	11	$1.50	4
1	2	13	$1.75	7
2	7	15	$2.00	10
3	12	17	$2.25	13
4	17	19	$2.50	16
5	22	21	$2.75	19
6	27	23	$3.00	22
7	32	25	$3.25	25
8	37	27	$3.50	28
9	42	29	$3.75	31

a. recursive: y(0) = -3; y(x+1)=y(x)+5 b. explicit: y = 11+2x c. recursive: f (0) = $1.50; f(t+1/8) = f(t) + $.25; explicit: f=$1.50 + $2.00t (where t is the number of miles, can also be set up in 1/8's of a mile as the other one) d. recursive: y(0)=11; y(x+1) = y(x)-1; explicit: y = 11 – x e. recursive: y(0)=4; y(x+1)=y(x)+3; explicit y=4+3x









5. One method is to note that the first seat in row r is one more than the last seat in row r-1, so we have $F = (r-1)^2 + 1 = r^2 - 2r + 2$.

6. In this case, the first seat is $F = r^2$, since $F = 1 + 3 + 5 + ...$, and we have shown earlier with pictures that odd numbers sum to perfect squares (the one in this equation refers to the first seat itself). We can use this equation to find $M = r^2 + r = r(r+1)$ and $L = r^2 + 2r = r(r+2)$. Note how nice these equations are!

7. and 8. a. exponential base 2 b. exponential base .1 c. not exponential d. exponential base 20

9. a. y(0)=12, y(x+1)=2y(x) b. b(0)=50, b(a+1) = .1 y(a) c. out (0) =1; out (in+1) = out (in) + 2^{in+1} d. q(0)=.06; q(p+1) = 20 q(p)

10. a. $y = 12 \cdot 2^x$ b. $b = 50 \cdot (.1)^a$ d. $q = .06 \cdot 20^p$