Name _____

Math 130
Spring 02

The Paper Number Game

This activity revisits something we did on the first day of class, although in a slightly different format. It includes both theoretical calculations and computer simulations. You may answer the questions directly on this sheet (or use extra paper). You do not have to write this activity as a paper. The activity is due by March 15 (the day before spring break). Note that Part 2 of your test will also be due that day.

This activity will be scored as follows:

Level 5 – Goes beyond a level 4 by doing suggested challenge problems, as well as everything for level 4.

Level 4 – Answers to questions are accurate, thorough, and thoughtful.

Level 3 – Answers to most questions are accurate, but reasoning doesn't always show thorough understanding or equivalent.

Level 2 – Shows good understanding of some parts of the assignment, but also shows a lot of misunderstanding.

Level 1 – Shows some understanding, but much more misunderstanding.

The Paper Number Game:

The Paper Number Game is played by n people, numbered $1, 2, 3, \ldots, n$. Each person writes his or her number on a slip of paper, the numbers are mixed thoroughly, and then each person randomly picks a slip of paper. Our initial question is, "How many people do we expect to pick their own numbers?" Throughout this activity, we will make this question more mathematically precise.

1. Let's start by making predictions about The Paper Number Game with 4, 10, 20, 100, and 1000 players. These predictions are just to get you thinking about the problem. Fill in the blanks below with numbers. Comment on any difficulties you had in deciding what to predict, but commit to some numbers anyway.

With 4 people playing, I think _____ people will pick their own names.

With 10 people playing, I think _____ people will pick their own names.

With 20 people playing, I think _____ people will pick their own names.

With 100 people playing, I think _____ people will pick their own names.

With 1000 people playing, I think _____ people will pick their own names.

2. Explain how you think the cases will differ when there are a small number or a large number of people.

3. Analyze the Paper Number Game when $n = 2$, i.e. when there are two players. What are the possible outcomes of the game? How likely are they? How many people would you expect to pick their own names? Explain. (You don't need to use precise mathematical language here.)

4. When there are 3 players in The Paper Name Game, we can make a table like the one below to show all possible outcomes of the game (i.e. the sample space). Note that this sample space is virtually identical to the one for the homework problem about randomly putting letters in envelopes:

Game #	Player 1's Pick	Player 2's Pick	Player 3's Pick	# Who Pick Their Own Names
1	1	2	3	3
2	1	3	2	1
3	2	1	3	1
4	2	3	1	0
5	3	1	2	0
6	3	2	1	1

Each horizontal line of this sample space is a possible game.

a. Explain how you could convince a skeptic that all possible games are represented.

Each of these six games is equally likely. Thus the theoretical probability for the number of people picking their own names in The Paper Name Game is as follows:

Probability that no one picks his or her own name: $\frac{2}{6} = \frac{1}{3} \approx 33.3\%$

Probability that exactly one person picks his or her own name: $\frac{3}{6} = \frac{1}{2} = 50\%$

Probability that exactly two people pick their own names: $\frac{0}{6} = 0 = 0\%$

Probability that all three people pick their own names: $\frac{1}{6} \approx 16.7\%$

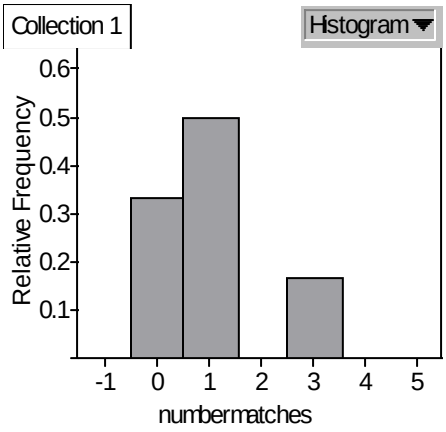
b. Explain how these probabilities were calculated.

c. Add up the four probabilities (as fractions). What do they add up to? What is significant about the sum?

d. Add up the four probabilities (as percentages). What do they add up to? What is significant about the sum?

e. Does the fraction $\frac{0}{6}$ correspond to the division problem $0 \div 6$ or does it correspond to $6 \div 0$? Explain why the fraction $\frac{6}{0}$ will never appear as a probability.

f. Below is a histogram for the theoretical probabilities for playing The Pick a Paper Number Game with 3 people. Explain how to interpret "Relative Frequency" and "numbermatches."



5. Now we will repeat a theoretical analysis for the case when four people play the game (this time you do all the work!)

a. Fill in the table below with each possible outcome of the game (i.e. with the sample space).

Game #	Player 1's Pick	Player 2's Pick	Player 3's Pick	Player 4's Pick	# Who Pick Own Names

b. Explain how you could convince a skeptic that all possible games are represented.

c. Fill in the blanks below for the game with four people:

Probability that no one picks his or her own name: _____ = _____ %

Probability that exactly one person picks his/her own name: _____ = _____ %

Probability that exactly two people pick their own names: _____ = _____ %

Probability that exactly three people pick their own names: _____ = _____ %

Probability that all four people pick their own names: _____ = _____ %

Explain briefly how you filled in the table:

d. Draw a histogram like the one in problem 4f to represent these probabilities.

e. What patterns do you notice in the games with 2, 3, and 4 players? Explain.

f. How do your predictions (in problem 1) compare with your theoretical results for the game with four people?

g. How many possible outcomes would there be in the sample space for the game played with five players? How could you convince a skeptic that your answer is correct? Note: you do not have to construct the sample space to answer this question. (Challenge – how many outcomes are there in the sample space for 6, 7, etc. players?)

If your work on the last question is accurate, it should convince you that figuring out the sample space and theoretical probability distribution for The Paper Number Game gets very tedious, very quickly. Thus, we are now going to switch to simulating the game on the computer.

There are two options for performing the simulations. Please try them both. The first is using a file I created in fathom called "paper number game simulation." If you don't have a copy of this file, please ask. Instructions for using the simulation are included in the file. To change the number of players, you must edit the case table near the top of the page. To simulate playing the game many times, you push the "collect more measures" button near the bottom of the page.

The second option is to use a java applet that can be found at: http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/bookapplets/chapter3/FixedPoints/FixedPoints.html

(There is no space in the word "bookapplets." If you don't feel like typing such a long address, do a search on Google for FixedPoints (with no space between the words), and the correct page should come up first.)

In the java applet, there are two numbers for you to fill in. The first is "number of permutations," which is the number of times you want to play the game. The second is "set size," which is the number of people playing the game. When you have entered both numbers, the probabilities of having zero to five people picking their own names ("fixed points) are shown.

The java applet is easier to use than the fathom simulation, although the fathom simulation gives more intermediate steps and more representations of the data so you can get a better sense of what's going on.

6. Begin by simulating the game with four players. What did you use for the simulations?

a. Do five separate runs of simulating ten games. Describe the most important similarities and differences between the data from each run. Be sure to include representations such as tables or graphs in your description.

b. Now do five separate runs of simulating 100 games, and report as in part a.

c. Do a simulation of a large number of games (at least 1000). Show the probabilities from the simulation of having 0, 1, 2, 3, or 4 people picking their own names. Give your data as percentages, and show the data both visually (a histogram) and in a table. Describe general features of the data in words (e.g. what is the most likely result, how much more likely is it than other results, what are the general trends, etc.)

d. Compare your results to the theoretical results you calculated in problem 5.

6. a. Simulate a large number (over 1000) of games with ten players. Describe the results below (as in 5c.)

b. How do these results compare to your prediction in question 1?

7. Simulate a large number of games with 20 players, report the results (as above), and compare to your earlier predictions.

8. Repeat for 100 players.

9. Repeat for 1000 players (if the computer is too slow, you can do fewer trials).

10. Summarize your findings. How do the probabilities change as the number of players increases? How did the simulations compare with the overall trends in your predictions? What is most surprising about your results?

11. Challenges (optional):

i. Explore further the effect of simulating with different numbers of trials (more or less than 1000).

ii. Explore more about the mean (average) number of people who pick their own name.

iii. Explore in more detail how the probability of no one picking his or her own names changes as the number of players increase. Explore for the probability of exactly 1 person picking his/her own name, for 2 people doing so, etc.

iv. Explore in fathom, trends in the largest number of people who pick their own names (i.e. the rightmost bar on the histogram). The java applet doesn't go beyond five people picking their own names, but it does happen.

v. Make up your own challenge.